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Fatou Theorems for Multi-index Bessel Series

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Abstract. We consider series defined by means of the generalized Bessel functions, find their domains of convergence and study the behaviour of such series on the boundaries of these domains. Analogues of the classical theorems for the power series like Cauchy-Hadamard, Abel, as well as Fatou type theorems are proposed.

Keywords: Bessel functions and generalizations, Series in multi-index Bessel functions, Cauchy-Hadamard, Abel and Fatou type theorems.
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INTRODUCTION

Let J_ν and C_ν denote respectively the classical Bessel function

$$J_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(k+\nu+1)}, \quad \nu \in \mathbb{C},\, z \in \mathbb{C}\setminus(-\infty,0], \tag{1}$$

and Bessel-Clifford function

$$C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu+k+1)}, \quad \nu \in \mathbb{C},\, z \in \mathbb{C}. \tag{2}$$

The function

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu+\mu k+1)} \qquad (\mu > -1,\, z \in \mathbb{C}), \tag{3}$$

that is a generalization of the Bessel-Clifford function, was introduced by Wright and called Bessel-Wright or misnamed as Bessel-Maitland function (after Sir Edward Maitland Wright). Initially, Wright defined (3) only for $\mu > 0$, and on a later stage extended its definition to $\mu > -1$ (see for example [2], [4]).
More general are the three- and four-index generalizations of the Bessel function J_ν, namely generalized Bessel-Maitland function:

$$J_{\nu,\lambda}^\mu(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(\nu+k\mu+\lambda+1)} \qquad (\mu > 0,\, \nu,\lambda \in \mathbb{C}, z \in \mathbb{C}\setminus(-\infty,0]), \tag{4}$$

introduced by Pathak (for details see [2], [4]), and the generalized Lommel-Wright function with 4 indices ($\mu > 0, q \in \mathbb{N}, \nu, \lambda \in \mathbb{C}$), introduced by de Oteiza, Kalla and Conde (for details see [2], [4]):

$$J_{\nu,\lambda}^{\mu,q}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q \Gamma(\nu+k\mu+\lambda+1)} \qquad (\mu > 0,\, q \in \mathbb{N},\, \nu, \lambda, \mathbb{C},\, z \in \mathbb{C}\setminus(-\infty,0])\cdot \tag{5}$$

Now, consider the families of Bessel-Maitland functions with integer indices ν, i.e. the Bessel-Maitland functions

$$J_n^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(n+\mu k+1)}, \quad n = 0,1,2,\ldots \tag{6}$$

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and the functions (4), (5) for indices of the kind $\nu = n-2\lambda$; $n = 0,1,2,...$, i.e. the generalized Bessel-Maitland function with 3 indices

$$J^{\mu}_{n-2\lambda,\lambda}(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda+k+1)\Gamma(n-\lambda+k\mu+1)}, \quad n = 0,1,2,\dots \tag{7}$$

and the generalized Lommel-Wright function with 4 indices

$$J^{\mu,q}_{n-2\lambda,\lambda}(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q\Gamma(n-\lambda+k\mu+1)}, \quad n = 0,1,2,\dots \tag{8}$$

In this paper we study series in such kind of functions and their convergence.

ASYMPTOTIC FORMULAE

The asymptotic formula with respect to the index

$$J^{\mu}_{n}(z) = \frac{1}{\Gamma(n+1)}(1+\theta^{\mu}_{n}(z)), \ z \in \mathbb{C}, \ \mu > 0; \quad \theta^{\mu}_{n}(z) \to 0 \ \text{as} \ n \to \infty \tag{9}$$

is given in [6] for the Bessel-Maitland functions. The functions $\theta^{\mu}_{n}(z)$ are entire functions. The convergence of $\left\{\theta^{\mu}_{n}(z)\right\}$ is uniform on the compact subsets of the complex plane $\mathbb{C}$. Considering explicitly $\theta^{\mu}_{n}(z)$, this result can be made sharper, as follows.

Lemma 1. *Let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists a constant C, $0 < C < \infty$, such that for each $n \in \mathbb{N}_0$ and each $z \in K$ the following inequality holds*

$$|\theta^{\mu}_{n}(z)| \le C\,\Gamma(n+1)/\Gamma(n+1+\mu).$$

Further, Stirling's formula gives that

$$\frac{\Gamma(n+1)}{\Gamma(n+1+\mu)} = O\left(\frac{1}{n^{\mu}}\right), \quad \text{for } n \in \mathbb{N}. \tag{10}$$

Consider now the generalized Lommel-Wright function with 4 indices for indices of the kind $\nu = n-2\lambda$; $n = 0,1,2,....$ Given a number λ, suppose that some coefficients in (8) equal zero, that is, there exist numbers $p \in \mathbb{N}_0$, $s \in \mathbb{N}$ such that the identity (8) can be written as

$$J^{\mu,\,q}_{n-2\lambda,\lambda}(z) = (z/2)^n \left(\frac{(-1)^p (z/2)^{2p}}{(\Gamma(\lambda+p+1))^q\,\Gamma(n-\lambda+p\mu+1)} + \sum_{k=p+s}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^q\,\Gamma(n-\lambda+k\mu+1)} \right). \tag{11}$$

It is not hard to verify that the function $J^{\mu,q}_{n-2\lambda,\lambda}$ is an entire function (see e.g. [3]).

About the functions (8), similar evaluations can be made. By means of these evaluations and using Stirling's formula (see e.g. [1]), one can obtain the following result (see for details [11]).

Lemma 2. *Let $\mu > 0$. Then the generalized Lommel-Wright functions* (8) *satisfy the following asymptotic formula*

$$J^{\mu,q}_{n-2\lambda,\lambda}(z) = \frac{(-1)^p (z/2)^{n+2p}}{(\Gamma(\lambda+p+1))^q\,\Gamma(n-\lambda+p\mu+1)}(1+\theta^{\mu,q}_{n-2\lambda,\lambda}(z)), \tag{12}$$

$$\theta^{\mu,q}_{n-2\lambda,\lambda}(z) \to 0 \ \textit{as} \ n \to \infty.$$

On the compact subsets of the complex plane $\mathbb{C}$ convergence is uniform and

$$\theta^{\mu,q}_{n-2\lambda,\lambda}(z) \ = \ O\left(\frac{1}{n^{s\mu}}\right), \quad (n \in \mathbb{N}) \tag{13}$$

with the corresponding $s \in \mathbb{N}$, depending on λ.

Remark 1. *By the way, having in view that the functions* (7) *can be obtained from* (8) *setting* $q = 1$*, the asymptotic formula for the generalized Bessel-Maitland functions follows just as a particular case of the above given.*

Remark 2. *The uniform convergence of* $\theta_n^{\mu}(z)$*,* $\theta_{n-2\lambda,\lambda}^{\mu}$ *and* $\theta_{n-2\lambda,\lambda}^{\mu,q}$ *on the compact subsets of* $\mathbb{C}$ *follows from Lemma 1 and formulae* (10)*,* (13)*, as well.*

SERIES IN GENERALIZED BESSEL FUNCTIONS

Multiplying the functions (6)–(8) with suitable coefficients and power functions, and taking into account the relation (11), we obtain a little bit modified accountable systems of functions as follows:

$$\widetilde{J}_n^{\mu}(z) = z^n \Gamma(n+1) J_n^{\mu}(z), \quad n = 0,1,2,\dots \tag{14}$$

$$\widetilde{J}_{n-2\lambda,\lambda}^{\mu}(z) = 2^{n+2p}\, \Gamma(\lambda+p+1)\Gamma(n-\lambda+p\mu+1) z^{-2p} J_{n-2\lambda,\lambda}^{\mu}(z), \quad n = 0,1,2,\dots \tag{15}$$

$$\widetilde{J}_{n-2\lambda,\lambda}^{\mu,q}(z) = 2^{n+2p} (\Gamma(\lambda+p+1))^q\, \Gamma(n-\lambda+p\mu+1) z^{-2p} J_{n-2\lambda,\lambda}^{\mu,q}(z), \quad n = 0,1,2,\dots \tag{16}$$

Remark 3. *According to the asymptotic formulae* (9) *and* (12) *it follows that there exists a natural number* N_0 *such that the functions* $\widetilde{J}_n^{\mu}(z)$*,* $\widetilde{J}_{n-2\lambda,\lambda}^{\mu}$ *and* $\widetilde{J}_{n-2\lambda,\lambda}^{\mu,q}$ *have no zeros for* $n > N_0$*, except for the origin.*

Remark 4. *Note that each of the functions* $\widetilde{J}_n^{\mu}(z)$*,* $\widetilde{J}_{n-2\lambda,\lambda}^{\mu}$ *and* $\widetilde{J}_{n-2\lambda,\lambda}^{\mu,q}$ $(n \in \mathbb{N}_0)$*, being an entire function, no identically zero, has no more than finite number of zeros in the closed and bounded set* $|z| \le R$ *(see [5], vol.1, ch. 3, § 6, 6.1, p. 305). Moreover, because of Remark 3, no more than finite number of these functions have some zeros, except for the origin.*

We consider the series in discussed above generalized Bessel functions in the complex plane and we briefly call them multi-index Bessel series. Namely, consider the series

$$\sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z), \quad \sum_{n=0}^{\infty} a_n \widetilde{J}_{n-2\lambda,\lambda}^{\mu}(z), \quad \sum_{n=0}^{\infty} a_n \widetilde{J}_{n-2\lambda,\lambda}^{\mu,q}(z) \qquad (\mu > 0,\ a_n, z \in \mathbb{C}), \tag{17}$$

and along with it - the power series

$$\sum_{n=0}^{\infty} a_n z^n.$$

In the process of studying the series (17), we obtain results analogical to the ones for the given power series. That is, it turns out that the series (17) have a behaviour like a power series.

CAUCHY-HADAMARD AND ABEL TYPE THEOREMS

In the beginning, we state a theorem of Cauchy-Hadamard type (proven in [7], [10]) and a corollary for the discussed series.

In what follows we use the notation $D(0;R)$ and $C(0;R)$ respectively for the open disk centered at the origin with a radius R and its boundary, i.e.:

$$D(0;R) = \{z : |z| < R,\ z \in \mathbb{C}\}, \quad C(0;R) = \partial D(0;R) = \{z : |z| = R,\ z \in \mathbb{C}\}.$$

Theorem 1 (of Cauchy-Hadamard type)**.** *The region of convergence of each series* (17) *with complex coefficients* a_n *is the disk* $D(0;R)$ *with a radius of convergence*

$$R = \left(\limsup_{n\to\infty} (\,|a_n|\,)^{1/n} \right)^{-1}. \tag{18}$$

More precisely, the series is absolutely convergent in the disk D(0;R) and divergent on the region $|z| > R$*. The cases* $R = 0$ *and* $R = \infty$ *fall in the general case.*

So, the considered series (17) converge in a respective disk, like in the theory of the widely used power series. Analogously, inside the corresponding disks, the convergence of the discussed series is uniform, i.e., the following corollary, similar to the classical Abel lema, holds.

Corollary 1.1. *Let any of the series* (17) *converges at the point* $z_0 \neq 0$. *Then it is absolutely convergent on the disk* $D(0;|z_0|)$. *Inside the disk* $D(0;R)$, *i.e. on each closed disk* $|z| \leq r < R$ (R *defined by* (14)), *the convergence is uniform.*

Proof. Indeed, since the considered series (e.g. the first of the series (17)) converges at the point $z_0 \neq 0$, then its radius of convergence R is a positive number, and moreover the point z_0 lies either in the disk $D(0;R)$ or on its boundary - the circle $C(0;R)$. That is why, the disk $D(0;|z_0|)$ is either a part of the domain of convergence or it coincides with it, whence the absolute convergence follows. To prove uniformity of the convergence inside the disk $D(0;R)$, it is sufficiently to show that the series is uniformly convergent on each closed disk $|z| \leq r$ ($r < R$). To this purpose, choosing a point ζ, $|\zeta| = \rho$, $r < \rho < R$ and considering the series (17), we estimate $|a_n \widetilde{J}_n^{\mu}(z)|$. First, mention that some of the values of $\widetilde{J}_n^{\mu}(\zeta)$, but only finite numbers of them, can be zero. Then, having in view (9), as well, it follows that there exists a natural number p such that the functions e $\widetilde{J}_n^{\mu}(z)$ have no zeros for $n > p$, except for the origin (see Remark 3). So, there exists a number p, such that the expression $|a_n \widetilde{J}_n^{\mu}(z)|$ can be written as follows

$$|a_n \widetilde{J}_n^{\mu}(z)| = |a_n \widetilde{J}_n^{\mu}(\zeta)| \frac{|\widetilde{J}_n^{\mu}(z)|}{|\widetilde{J}_n^{\mu}(\zeta)|} = |a_n \widetilde{J}_n^{\mu}(\zeta)| \frac{|z^n||1+\theta_n^{\mu}(z)|}{|\zeta^n||1+\theta_n^{\mu}(\zeta)|} \leq |a_n \widetilde{J}_n^{\mu}(\zeta)| \frac{|1+\theta_n^{\mu}(z)|}{|1+\theta_n^{\mu}(\zeta)|}$$

for all $n > p$ and $|z| \leq r$.

Because of Lemma 1 and the relation (10), taking into account that $\lim_{n\to\infty} \frac{1}{n^{\mu}} = 0$, we obtain the equalities $\lim_{n\to\infty}(1 + \theta_n^{\mu}(z)) = 1$, $\lim_{n\to\infty}(1+\theta_n^{\mu}(\zeta))^{-1} = 1$. Therefore, there exist numbers A and B such that $|1+\theta_n^{\mu}(z)||1+\theta_n^{\mu}(\zeta)|^{-1} \leq AB$ and hence $|a_n \widetilde{J}_n^{\mu}(z)| \leq AB|a_n \widetilde{J}_n^{\mu}(\zeta)|$, for all the values of $n > p$ and $|z| \leq r$. Since the series $\sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(\zeta)$ is absolutely convergent and by the Weierstrass criterium for the uniform convergence, the proof is completed. For the other two cases the proofs go analogously. □

The very disk of convergence is not obligatory a domain of uniform convergence and on its boundary the series may even be divergent.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_{φ} be an arbitrary angular domain with size $2\varphi < \pi$ and with a vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0 and d_{φ} be the part of the angular domain g_{φ}, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle. The next inequality can be verified inside d_{φ}

$$|z - z_0| \cos\varphi < 2(|z_0| - |z|). \tag{19}$$

The following theorem refers to the uniform convergence of the series (17) on the set d_{φ} and the limit of its sum at the point z_0, provided $z \in D(0;R) \cap g_{\varphi}$.

Theorem 2 (of Abel type)**.** *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers, R be the real number defined by* (18) *and* $0 < R < \infty$. *If* $\widetilde{f}(z;\mu)$ (*respectively* $\widetilde{g}(z;\lambda,\mu)$ *or* $\widetilde{h}(z;\lambda,\mu,q)$) *is the sum of the series* (17) *on the domain* $D(0;R)$, *i.e.*

$$\widetilde{f}(z;\mu) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z), \quad z \in D(0;R)$$

$$\text{(respectively } \widetilde{g}(z;\lambda,\mu) = \sum_{n=0}^{\infty} a_n \widetilde{J}_{n,\lambda}^{\mu}(z), \quad \text{or } \widetilde{h}(z;\lambda,\mu,q) = \sum_{n=0}^{\infty} a_n \widetilde{J}_{n,\lambda}^{\mu,q}(z))$$

and this series converges at the point z_0 *of the boundary* $C(0;R)$, *then:*

(i) *The following relation holds*

$$\lim_{z\to z_0} \widetilde{f}(z;\mu) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z_0),$$

$$\text{(respectivelly } \lim_{z\to z_0} \widetilde{g}(z;\lambda,\mu) = \sum_{n=0}^{\infty} a_n \widetilde{J}^{\mu}_{n,\lambda}(z_0) \quad \text{or} \quad \lim_{z\to z_0} \widetilde{h}(z;\lambda,\mu,q) = \sum_{n=0}^{\infty} a_n \widetilde{J}^{\mu,q}_{n,\lambda}(z_0)),$$

provided $z \in D(0;R) \cap g_{\varphi}$.

(ii) *The series* (17) *is uniformly convergent on the domain* d_{φ}.

Proof. The results, referring to the limits in (i), are obtained in [7]–[10]. To prove the uniform convergence in (ii), we use the inequality (19) that is the crucial point of the proof.

So, let $z \in d_{\varphi}$. Setting

$$S_k(z) = \sum_{n=0}^{k} a_n \widetilde{J}^{\mu}_n(z), \quad S_k(z_0) = \sum_{n=0}^{k} a_n \widetilde{J}^{\mu}_n(z_0), \quad \lim_{k\to\infty} S_k(z_0) = s, \tag{20}$$

$$\beta_n = S_n(z_0) - s, \quad \beta_n - \beta_{n-1} = a_n \widetilde{J}^{\mu}_n(z_0),$$

we obtain

$$S_{k+p}(z) - S_k(z) = \sum_{n=0}^{k+p} a_n \widetilde{J}^{\mu}_n(z) - \sum_{n=0}^{k} a_n \widetilde{J}^{\mu}_n(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{J}^{\mu}_n(z).$$

According to Remark 3, there exists a natural number N_0 such that $\widetilde{J}^{\mu}_n(z_0) \neq 0$ when $n > N_0$. Let $k > N_0$ and $p > 0$. Then, using the denotation

$$\gamma_n(z;z_0) = \widetilde{J}^{\mu}_n(z)/\widetilde{J}^{\mu}_n(z_0),$$

we can write the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{J}^{\mu}_n(z_0) \frac{\widetilde{J}^{\mu}_n(z)}{\widetilde{J}^{\mu}_n(z_0)} = \sum_{n=k+1}^{k+p} a_n \widetilde{J}^{\mu}_n(z_0) \gamma_n(z;z_0).$$

Now, by the Abel transformation (see in [5], vol.1, ch.1, p.32, 3.4:7), we obtain consecutively:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1}) \gamma_n(z;z_0)$$

$$= \beta_{k+p}\gamma_{k+p}(z;z_0) - \beta_k \gamma_{k+1}(z;z_0) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z;z_0) - \gamma_n(z;z_0)),$$

$$S_{k+p}(z) - S_k(z) = (S_{k+p}(z_0) - s)\gamma_{k+p}(z;z_0) - (S_k(z_0) - s)\gamma_{k+1}(z;z_0)$$
$$+ \sum_{n=k+1}^{k+p-1} (S_n(z_0) - s) \times \left(\frac{\widetilde{J}^{\mu}_n(z)}{\widetilde{J}^{\mu}_n(z_0)} - \frac{\widetilde{J}^{\mu}_{n+1}(z)}{\widetilde{J}^{\mu}_{n+1}(z_0)} \right).$$

So, using last relation, we are going to estimate the module of the difference $S_{k+p}(z) - S_k(z)$ as follows:

$$|S_{k+p}(z) - S_k(z)| \le |S_{k+p}(z_0) - s||\gamma_{k+p}(z;z_0)| + |S_k(z_0) - s||\gamma_{k+1}(z;z_0)|$$
$$+ \sum_{n=k+1}^{k+p-1} |S_n(z_0) - s| \times \left| \frac{\widetilde{J}^{\mu}_n(z)}{\widetilde{J}^{\mu}_n(z_0)} - \frac{\widetilde{J}^{\mu}_{n+1}(z)}{\widetilde{J}^{\mu}_{n+1}(z_0)} \right|. \tag{21}$$

Because of (9) and the relations $\lim_{n\to\infty} \frac{1}{n^{\mu}} = 0$, $\lim_{n\to\infty}(1+\theta^{\mu}_n(z_0))^{-1} = 1$, there exist numbers A and $N_1 > N_0$ such that $|1+\theta^{\mu}_n(z)| \le A/2$ for all the natural values of n and $|1+\theta^{\mu}_n(\zeta)|^{-1} < 2$ for $n > N_1$, whence

$$|\gamma_n(z,z_0)| \le A \quad \text{for } n > N_1. \tag{22}$$

Further, setting

$$j_n(z,z_0) = \frac{\widetilde{J}^{\mu}_n(z)}{\widetilde{J}^{\mu}_n(z_0)} - \frac{\widetilde{J}^{\mu}_{n+1}(z)}{\widetilde{J}^{\mu}_{n+1}(z_0)} = \frac{z^n}{z_0^n} \times \left(\frac{1+\theta^{\mu}_n(z)}{1+\theta^{\mu}_n(z_0)} - \frac{z}{z_0} \times \frac{1+\theta^{\mu}_{n+1}(z)}{1+\theta^{\mu}_{n+1}(z_0)} \right)$$

and observing that $j_n(z_0,z_0)=0$, we apply the Schwarz lemma, named after Hermann Amandus Schwarz, for $j_n(z,z_0)$. So, we get that there exists a constant C:

$$|j_n(z,z_0)| = |\widetilde{J}_n^\mu(z)/\widetilde{J}_n^\mu(z_0) - \widetilde{J}_{n+1}^\mu(z)/\widetilde{J}_{n+1}^\mu(z_0)| \le C|z-z_0||z/z_0|^n,$$

whence, and in accordance with (19):

$$\sum_{n=k+1}^{k+p+1} |j_n(z,z_0)| \le \sum_{n=0}^{\infty} C|z-z_0||z/z_0|^n = C|z_0| \times \frac{|z-z_0|}{|z_0|-|z|} < \frac{2C|z_0|}{\cos\varphi}. \tag{23}$$

Let ε be an arbitrary positive number. Taking in view the third of the relations (20), we can confirm that there exists a positive number $N_2 > N_0$ so large that

$$|S_n(z_0) - s| < \min\left(\frac{\varepsilon}{3A}, \frac{\varepsilon\cos\varphi}{6C|z_0|}\right) \quad \text{for } n > N_2. \tag{24}$$

Now, let $N = N(\varepsilon) = \max(N_1, N_2)$ and $k > N$. Therefore (21)–(23) give

$$|S_{k+p}(z) - S_k(z)| < \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|} \times \sum_{n=k+1}^{k+p+1} |j_n(z,z_0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|} \times \frac{2C|z_0|}{\cos\varphi} = \varepsilon,$$

that completes the proof of the theorem for the considered series.

The uniform convergence of the other considered series go in the same way. In fact, the result for the second series is just a particular case of the third one for $q=1$. □

FATOU TYPE THEOREM

Let us consider the power series $\sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n and a radius of convergence $0 < R < \infty$, $f(z)$ be the sum of this series on the disk of convergence $D(0;R)$ i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D(0;R). \tag{25}$$

and $z_0 \in \partial D(0;R)$ be a regular point for f. Then the series (25) may be convergent or divergent at z_0. Practically, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points. For example: The power series $\sum_{n=0}^{\infty} z^n$ is divergent at each point of the unit circle $C(0;1)$ regardless of the fact that all the points of this circle, except for $z=1$, are regular for its sum. The series $\sum_{n=1}^{\infty} n^{-2} z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z=1$, is a singular (i.e. not regular) for its sum. However, under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exist (see for details Fatou theorem in [5], Vol.1, Ch. 3, §7, 7.3, p.357). Namely, the following theorem holds true.

Theorem 3 (of Fatou). *If the coefficients of the power series with the unit disk of convergence tend to the zero, i.e.* $\lim_{n\to\infty} a_n = 0$, *then the power series converges, even uniformly, on each arc of the unit circle, all the points of which (including the ends of the arc) are regular for the sum of the series.*

Propositions referring to the properties discussed above have been also established for series in other special functions, e.g. in the Laguerre and Hermite polynomials, as well as in Bessel and Mittag-Leffler systems (see [14], resp. [12], [13]). Here we give such a type of theorem for the generalized Bessel systems as follows.

Theorem 4 (of Fatou type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the conditions*

$$\lim_{n\to\infty} a_n = 0, \qquad \limsup_{n\to\infty} (\,|a_n|\,)^{1/n} = 1, \tag{26}$$

and $F(z)$ be the sum of the series (17) *on the unit disk $D(0;1)$, i.e.*

$$F(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_n^{\mu}(z), \qquad z \in D(0;1).$$

$$(resp.:\ F(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_{n,\lambda}^{\mu}(z), \quad or\ F(z) = \sum_{n=0}^{\infty} a_n \widetilde{J}_{n,\lambda}^{\mu;q}(z), \qquad z \in D(0;1)).$$

Let σ be an arbitrary arc of the unit circle $C(0;1)$ with all its points (including the ends) regular to the function F. Then the series (17) *converges, even uniformly, on the arc σ.*

Proof. We prove the theorem for the first of the series (17). Since all the points of the arc σ are regular to the function $F(z)$ there exists a region $G \supset \sigma$ where the function F can be continued. Denoting $\widetilde{G} = G \cup D(0;1)$, we define the function ψ in the region $\widetilde{G}$ by the equality

$$\psi(z) = F(z), \quad z \in D(0;1).$$

More precisely, it means that ψ is a single valued analytical continuation of F in the domain $\widetilde{G}$.

Let $\rho > 0$ be the distance between the boundary $\partial\widetilde{G}$ of the region $\widetilde{G}$ and the arc σ ($\partial\widetilde{G}$ contains a part of the unit circle $|z| = 1$), and take the points ζ_1, ζ_2

$$\zeta_1, \zeta_2 \notin \sigma, \quad |\zeta_1| = |\zeta_2| = 1,$$

such that the distances between each of the points ζ_1, ζ_2 and the respective closer end of the arc σ are equal to $\rho/2$, and

$$z_1 = \zeta_1(1+\rho/2), \quad z_2 = \zeta_2(1+\rho/2).$$

Define the auxiliary function

$$\varphi_n(z) = \psi(z) - \sum_{k=0}^{n} a_k \widetilde{J}_k^{\mu}(z) \tag{27}$$

and note that, according to Remark 3, there exists a natural number N_0 such that $\widetilde{J}_n^{\mu}(z) \neq 0$ when $n > N_0$. Now, letting $n \geq N_0$, we introduce the notation

$$\omega_n(z) = \frac{\varphi_n(z)}{\widetilde{J}_{n+1}^{\mu}(z)}(z-\zeta_1)(z-\zeta_2). \tag{28}$$

In order to prove that the sequence $\left\{\sum_{k=0}^{n} a_k \widetilde{J}_k^{\mu}(z)\right\}$ is uniformly convergent on the arc σ, it is sufficiently to show that the sequence $\{\omega_n(z)\}_{n=N_0}^{\infty}$ tends uniformly to zero on the boundary $\partial\Delta$ of the sector $\Delta = Oz_1z_2$ which is a compact set and after that estimate $\varphi_n(z)$ on the arc σ.

To this end, we come back to Lemma 1 and the relation (10). Just mention that since $\lim_{n\to\infty} \frac{1}{n^{\mu}} = 0$, there exist numbers C and $\widetilde{N} > N_0$ such that $|1+\theta_n(z)| \leq C/2$ for all the values of $n \in \mathbb{N}$ and $1/2 \leq |1+\theta_n(z)| \leq 2$ for $n > \widetilde{N}$ on an arbitrary compact subset of $\mathbb{C}$.

Now, taking $\varepsilon > 0$ and setting

$$R = 1+\rho/2, \quad \varepsilon_1 = \frac{\varepsilon\rho^3}{8(8CR^2+\rho)}, \quad M = \max_{z\in[\Delta]} |\psi(z)| \qquad ([\Delta] = \Delta \cup \partial\Delta),$$

we have to consider four cases as follows.

1) First, let $z \in (O,\zeta_1) \cup (O,\zeta_2) \subset D(0;1)$.

In the unit disk, according to (27), we have consecutively:

$$\omega_n(z) = \sum_{k=0}^{\infty} a_{n+k+1} \frac{\widetilde{J}^{\mu}_{n+k+1}(z)}{\widetilde{J}^{\mu}_{n+1}(z)} (z-\zeta_1)(z-\zeta_2),$$

$$\omega_n(z) = \sum_{k=0}^{\infty} a_{n+k+1}\, z^k \,\frac{(1+\theta^{\mu}_{n+k+1}(z))}{(1+\theta^{\mu}_{n+1}(z))} (z-\zeta_1)(z-\zeta_2). \tag{29}$$

Since $a_n \to 0$, there exists a number $N_1 = N_1(\varepsilon_1) > \widetilde{N}$, such that

$$|\omega_n(z)| < \varepsilon_1 \sum_{k=0}^{\infty} |z|^k \left| \frac{(1+\theta^{\mu}_{n+k+1}(z))}{(1+\theta^{\mu}_{n+1}(z))} \right| |(z-\zeta_1)||(z-\zeta_2)|$$

$$< 2C\varepsilon_1 \sum_{k=0}^{\infty} |z|^k (1-|z|) = 2C\varepsilon_1$$

for $n > N_1$, i.e.

$$|\omega_n(z)| < 2C\varepsilon_1. \tag{30}$$

2) $z \in (\zeta_1, z_1) \cup (\zeta_2, z_2)$.

In this case $|z-\zeta_1| = |z|-1$, $|z-\zeta_2| \le |z| + |\zeta_2| < 2R$, and taking into account (9) and (27) we can write the following inequalities for the absolute value of $\omega_n(z)$

$$\omega_n(z) = \frac{\psi(z) - \sum_{k=0}^{n} a_k\, z^k\, (1+\theta^{\mu}_k(z))}{z^{n+1}(1+\theta^{\mu}_{n+1}(z))} (z-\zeta_1)(z-\zeta_2),$$

namely

$$|\omega_n(z)| \le \frac{M + \sum_{k=0}^{n} |a_k||z|^k |(1+\theta^{\mu}_k(z))|}{|z|^{n+1}|(1+\theta^{\mu}_{n+1}(z))|}\, 2R(|z|-1)$$

$$< 2R\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right) \frac{(|z|-1)}{|z|^{n+1}} + 2\varepsilon_1 RC\, \frac{(|z|-1)}{|z|^{n+1}} \sum_{k=N_1+1}^{n} |z|^k\,.$$

Furthermore, having in mind that, on the one hand:

$$\frac{(|z|-1)}{|z|^{n+1}} < \frac{(|z|-1)}{|z|^{n+1}-1} = \frac{1}{|z|^n + \cdots + 1} < \frac{1}{n+1},$$

and on the other hand:

$$\sum_{k=N_1+1}^{n} |z|^k = \frac{|z|^{n+1} - |z|^{N_1+1}}{(|z|-1)} < \frac{|z|^{n+1}}{(|z|-1)},$$

we conclude that

$$|\omega_n(z)| < \frac{2R}{n+1}\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right) + 2\varepsilon_1 RC.$$

Then, since $n^{-1} \to 0$, there exists a number $N_2 = N_2(\varepsilon_1) > N_1$ such that

$$\frac{2R}{n+1}\left(2M + \sum_{k=0}^{N_1} C|a_k|R^k\right) < \varepsilon_1$$

for $n > N_2$, i.e.

$$|\omega_n(z)| < (1+2RC)\varepsilon_1. \tag{31}$$

3) z belongs to the arc $\widehat{z_1 z_2}$ (including the ends).

Then $|z-\zeta_1| < 2R$, $|z-\zeta_2| < 2R$ and hence

$$|\omega_n(z)| < \frac{4R^2\left(2M+\sum\limits_{k=0}^{n}C|a_k|R^k\right)}{R^{n+1}} < \frac{4\left(2M+\sum\limits_{k=0}^{N_1}C|a_k|R^k\right)}{R^{n-1}} + \frac{4\varepsilon_1 C\left(\sum\limits_{k=N_1+1}^{n}R^k\right)}{R^{n-1}}$$

$$< \frac{4\left(2M+\sum\limits_{k=0}^{N_1}C|a_k|R^k\right)}{R^{n-1}} + \frac{8\varepsilon_1 CR^2}{\rho}.$$

Since $R^{-n}\to 0$, there exists a number $N_3 = N_3(\varepsilon_1) > N_1$, such that

$$|\omega_n(z)| < \left(\frac{8CR^2}{\rho}+1\right)\varepsilon_1 \tag{32}$$

for $n > N_3$.

4) $z \in \{O, \zeta_1, \zeta_2\}$.

In this case we have $\omega_n(0) = a_{n+1}\zeta_1\zeta_2$, whence $|\omega_n(0)| = |a_{n+1}| < \varepsilon_1$ for $n > N_1$, and $\omega_n(\zeta_{1,2}) = 0$.

Let $N = \max\{N_1, N_2, N_3\}$ and $n > N$, then having in view the inequalities (30) – (32), we can write on the boundary of the region Δ:

$$|\omega_n(z)| < \max\left(2C\varepsilon_1, (2RC+1)\varepsilon_1, \left(\frac{8CR^2}{\rho}+1\right)\varepsilon_1\right) = \left(\frac{8CR^2}{\rho}+1\right)\varepsilon_1.$$

Hence according to the principle of the maximum of the modulus

$$|\omega_n(z)| < \left(\frac{8CR^2+\rho}{\rho}\right)\varepsilon_1, \quad z\in\sigma. \tag{33}$$

Eventually, according to (9), (27) and (29), since $|z| = 1$ on the arc σ,

$$|\omega_n(z)| = \frac{\left|\psi(z)-\sum\limits_{k=0}^{n}a_k\widetilde{J}_k^{\mu}(z)\right|}{|z^{n+1}||1+\theta_{n+1}^{\mu}(z)|}|z-\zeta_1||z-\zeta_2| > \frac{1}{2}\cdot\frac{\rho^2}{4}\left|\psi(z)-\sum_{k=0}^{n}a_k\widetilde{J}_k^{\mu}(z)\right|, \tag{34}$$

whence applying the inequality (33), the relation (34) yields to

$$\left|\psi(z)-\sum_{k=0}^{n}a_k\widetilde{J}_k^{\mu}(z)\right| < \frac{8}{\rho^2}|\omega_n(z)| < \frac{8\varepsilon_1}{\rho^3}\left(8CR^2+\rho\right) = \varepsilon, \quad z\in\sigma,$$

that proves the theorem in this case.

The proof for the third series goes analogously, and for the second - is a particular case of the last one, taking $q = 1$. $\square$

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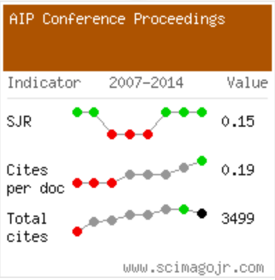

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